Substitution Formulas for Laplace Transformations*

By R. G. Buschman

In the literature there are numerous scattered formulas for special cases of the Laplace transforms of composite functions; i.e., for $\mathfrak{L}\{k(t)F[g(t)]\}$ involving $\mathfrak{L}\{F(t)\}$ explicitly. Formulas of this type are useful, for example, in extending existing tables. In the tables of Erdélyi, et al., [4] some "general formulas" are listed, 4.1(22)–(27). Further formulas of this type appear in many places; for example, the tables of G. Doetsch [3], N. W. McLachlan and P. Humbert [5], N. W. McLachlan, P. Humbert and L. Poli [6], and B. van der Pol and H. Bremmer [7]. Almost all of these formulas fall, with the introduction of parameters, under three cases:

$$\mathfrak{L}\{t^c F(at^b)\}\,, \quad \text{for} \quad b < 0\,, \quad c > 1 \text{ or } b > 1\,, \quad c < 1\,; \quad a > 0\,; \tag{1}$$

$$\mathfrak{L}\{(e^{bt} - 1)^c F[a(e^{bt} - 1)]\}\,, \quad \text{for} \quad a, b > 0\,; \tag{2}$$

$$\mathfrak{L}\{\cosh^k tF(a \sinh t)\}\,, \quad \text{for} \quad a > 0\,, \quad k \geqq 0 \quad \text{and an integer.} \tag{3}$$

The general form of the expression is

$$\mathfrak{L}\{k(t)F[g(t)]\} = \int_0^\infty \Phi(s, u) f(u) du \tag{4}$$

with $\mathfrak{L}\{F\} = f$ and Φ dependent on k and g.

A few others are known, such as

$$\mathfrak{L}\{F(\cosh t - 1)\}\,, \qquad \text{see [5]}\,; \tag{5}$$

$$\mathfrak{L}\{(t + b/a)^c F(at^2 + 2bt)\}\,, \quad \text{for} \quad a, b > 0\,, \qquad \text{see [1]}\,. \tag{6}$$

Compositions which involve combinations in pairs of these substitutions could in general be expressed in terms of an iterated integral representation, but certain of these relations can be reduced to a single integral representation. A combination of (1) and (2) leads to $F[a/(e^{bt} - 1)]$ which appears in the table. Since the power function, when used as the multiplier function k, leads to partial differentiation of the function Φ with respect to s, additional formulas can thus be obtained from this combination. Further, if the multiplier function k is the exponential function or some combination of it, such as powers of the hyperbolic sine or hyperbolic cosine, we are led to shifting, differencing, and other such operations applied to the function Φ.

In the accompanying table we list some additional formulas which do not fall under the cases (1)–(3); we state merely k, g, and Φ, inasmuch as the formulas are all of the type of (4). The method of derivation is along the lines of that used in [1] and [2].

It is of interest to note the gap, $0 < b < 1$, in (1) where an open problem exists.

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In particular, the problem of $\mathfrak{L}\{F(t^{1/2})\}$ occurs, whereas the case of $\mathfrak{L}\{F(t^2)\}$ is covered. Of peculiar note with respect to this open question are the pairs involving combination formulas as (2), (7); (3), (5); and (6), (12).

We define the unit function, u, by $u(x) = 0, x < 0; = 1, x \geqq 0$, and use the notations as in *Tables of Integral Transforms* [4].

TABLE

$k(t)$	$g(t)$
$\Phi(s, u)$	

(7) 1 $\qquad a(e^{bt} - 1)^{1/2}$

$[2\pi^{1/2}/s\Gamma(s/b)]\,(au/2)^{s/b+1/2}J_{s/b-1/2}(au)$

$(a, b > 0)$

(8) $(e^{bt} - 1)^c$ $\qquad a(e^{bt} - 1)^{-1}$

$[b\Gamma(c + 2)]^{-1}(au)^{c/2}e^{-au/2}M_{s/b-c/2,c/2+1/2}(au)$

$(a, b > 0\,;\, \mathrm{Re}\, c > -2)$

(9) $(t + c)^{-\nu-1}$ $\qquad \frac{1}{2}at^2/(t + c)$

$(auc^2)^{-\nu/2}(au - 2s)^{\nu/2}e^{-c(au-s)}I_\nu[c(a^2u^2 - 2sau)^{1/2}]U(au - 2s)$

$(\mathrm{Re}\,\nu > -1\,;\quad a, c > 0)$

(10) $(t + c)^{-\nu-1}$ $\qquad \frac{1}{2}t(t + 2c)/(t + c)$

$e^{cs}(uc^2)^{-\nu/2}(u - 2s)^{\nu/2}J_\nu[c(u^2 - 2su)^{1/2}]U(u - 2s)$

$(\mathrm{Re}\,\nu > -1\,;\quad c > 0)$

(11) $(t + b)^{-1/2}[(t^2 + b^2)^{1/2} + t]^{-\nu}$ $\qquad a[(t^2 + b^2)^{1/2} - b]$

$b^{-\nu}e^{-abu}[(au - s)/(au + s)]^{\nu/2}I_\nu[b(a^2u^2 - s^2)^{1/2}]U(au - s)$

$(\mathrm{Re}\,\nu > -1\,;\quad a, b > 0)$

(12) $(t^2 + 2bt)^{-1/2}[t + b + (t^2 + 2bt)^{1/2}]^{-\nu}$ $\qquad a(t^2 + 2bt)^{1/2}$

$b^{-\nu}e^{bs}[(au - s)/(au + s)]^{\nu/2}J_\nu[b(a^2u^2 - s^2)^{1/2}]U(au - s)$

$(\mathrm{Re}\,\nu > -1\,;\quad a, b > 0)$

(13) 1 $\qquad a[(e^{bt} + c^2)^{1/2} - (1 + c^2)^{1/2}]$

$[2c\pi^{1/2}/s\Gamma(s/b)]\exp\,[-au(1 + c^2)^{1/2}](au/2c)^{s/b+1/2}I_{s/b-1/2}(acu)$

$(a, b, c > 0)$

(14) $(e^{bt} + c^2)^{-1/2}$ $\qquad$ Ditto

$[2\pi^{1/2}/s\Gamma(s/b)]\exp\,[au(1 + c^2)^{1/2}](au/2c)^{s/b+1/2}I_{s/b+1/2}(acu)$

$(a, b, c > 0)$

(15) 1 $\qquad$ $a[t + b - (t + b^2)^{1/2}], \quad b = c + \frac{1}{2}$

$\frac{1}{2}\pi^{-1/2}u(au - s)^{-3/2} \exp [cs - c^2(au - s) - s^2/4(au - s)]U(au - s)$

$(a, c > 0)$

(16) $(t + b)^{-1/2}[(t^2 + b^2)^{1/2} - 1/2]^{2\nu}$ $\qquad$ Ditto

$2^{-\nu}\pi^{-1/2}(au - s)^{-\nu-1/2} \exp [cs - c^2(au - s) - s^2/8(au - s)]$

$(a, c > 0)$ $\qquad$ $D_{2\nu}\{s/[2(au - s)]^{1/2}\}U(au - s)$

(17) Ditto, $\nu = 0$ $\qquad$ Ditto

$\pi^{-1/2}(au - s)^{-1/2} \exp [cs - c^2(au - s) - s^2/4(au - s)]U(au - s)$

$(a, c > 0)$

(18) Ditto, $\nu = n/2$ $\qquad$ Ditto

$(2^n\pi)^{-1/2}(au - s)^{-n/2-1/2} \exp [cs - c^2(au - s) - s^2/4(au - s)]$

$(a, c, n > 0)$ $\qquad$ $He_n\{s/[2(au - s)]^{1/2}\}U(au - s)$

(19) Ditto, $\nu = \frac{1}{2}$ $\qquad$ Ditto

$\frac{1}{2}\pi^{-1/2}s(au - s)^{-3/2} \exp [cs - c^2(au - s) - s^2/4(au - s)]U(au - s)$

$(a, c > 0)$

(20) Ditto, $\nu = -\frac{1}{2}$ $\qquad$ Ditto

$\exp [cs - c^2(au - s)] \operatorname{Erfc} \{s/[2(au - s)]^{1/2}\}U(au - s)$

$(a, c > 0)$

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